To study $r_D(n)$ (or $r_Q(n)$) it is natural to study the Dirichlet series $\sum_{n\geq 1}^{\infty} \frac{r_D(n)}{n^s}$ $\left(\sum \frac{r_Q(n)}{n^s}\right.$

We want to relate this series to $L(s, \chi_D)$ for some character mod D.

Recall A Dirichlet character χ mod N is a character of the group $(\mathbb{Z}/N\mathbb{Z})^*$

$$(\mathbb{Z}/N\mathbb{Z})^* = \{ n \bmod N \mid (n,N) = 1 \}.$$

For such a character we use again χ to denote $\chi : \mathbb{Z} \to \mathbb{C}$ defined via

$$\chi(n) = \begin{cases} \chi(n \bmod N) & \text{if } (n,N)=1 \\ 0 & \text{otherwise} \end{cases}$$

We first recall

Defn Let p odd prime. The Legendre symbol

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } a = \square \bmod p, \text{ ie } a = x^2 (p) \text{ for some} \\ -1 & \text{if } a \neq \square \bmod p \\ 0 & \text{if } a \equiv 0 \bmod p. \end{cases}$$

Legendre symbol is a charac. mod p, ie multiplicative in the numerator $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$ and $a \equiv b \bmod p \Rightarrow \left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$

Quadratic reciprocity p, q distinct odd primes

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)}$$

Legendre symbol is a specialization of Jacobi symbol

for n odd, positive $n = p_1^{e_1} \cdots p_s^{e_s}$

The Jacobi symbol $\left(\frac{a}{n}\right) := \left(\frac{a}{p_1}\right)^{e_1} \cdots \left(\frac{a}{p_s}\right)^{e_s}$

Jacobi symbol is defined only when the numerator an integer, denominator a positive odd integer. $\left(\frac{a}{n}\right)$ is a Dirichlet char. mod n. It is also multiplicative in the denominator but no longer keeps track of squares

eg $\left(\frac{7}{65}\right) = \left(\frac{7}{5}\right)\left(\frac{7}{13}\right) = (-1)(-1) = 1$ but 7 is not a square mod 65.

For odd positive d and n quadratic reciprocity holds: $\left(\frac{d}{n}\right)\left(\frac{n}{d}\right) = (-1)^{\frac{n-1}{2}\frac{d-1}{2}}$

Next we define the Kronecker symbol modulo D, where D is a fund. discriminant.

Recall: An integer D is called a fundamental disc if either $D \equiv 1 \bmod 4$, D squarefree

or $D \equiv 0 \bmod 4$, $D/4$ squarefree, $\frac{D}{4} \equiv 2$ or 3 (4)

For a fund. disc D we define the Kronecker symbol mod D as

$$\chi_D(p) := \left(\frac{D}{p}\right) \quad \text{if } p \text{ odd prime}$$

$$\chi_D(2) := \begin{cases} 0 & \text{if } D \equiv 0 \ (4) \\ +1 & \text{if } D \equiv 1 \ (8) \\ -1 & \text{if } D \equiv 5 \ (8) \end{cases}$$

$$\chi_D(p_1^{e_1} \cdots p_k^{e_k}) = \chi_D(p_1)^{e_1} \chi_D(p_2)^{e_2} \cdots \chi_D(p_k)^{e_k}$$

(χ_1 is the trivial character $\chi_1(a) = 1 \ \forall a \in \mathbb{Z}$.

The function $n \to \chi_D(n)$ is periodic mod $|D|$. It is a Dirichlet character mod $|D|$. $\chi_D(-1) = \begin{cases} 1 & \text{if } D > 0 \\ -1 & \text{if } D < 0 \end{cases}$

(χ_D is primitive ie its conductor is equal to its modulus.

conductor is the smallest non-zero period of a character.)

We have now the following multiplicative description of $r_D(n)$

<u>Thm 7-8</u> Let $D<0$ fundamental
Then

$$r_D(n) = \begin{cases} 0 & \text{if } p^2 \mid n \text{ for some } p \mid D \\ \prod_{p \mid n} (1 + \chi_D(p)) & \text{otherwise} \end{cases}$$

<u>Pf.</u> ~~Exercise~~ (Skip if no time or first do 7.9)

(1) First note that $r_D(n)$ is indeed 0 if for some $p \mid D$, $p^2 \mid n$. Suppose $r_D(n) \neq 0$
If p is odd and $b^2 \equiv D \bmod 4n$ has a soln
Then $b^2 \equiv D \bmod p^2$ (since $p^2 \mid n$)
Hence $b^2 = D + kp^2$ for some k
But then since $p \mid D$, $p \mid b^2$. Hence $p^2 \mid b^2$
This then implies $p^2 \mid D$. But this cannot happen since D is fundamental

If $2 \mid D$, and $4 \mid n$ then we get
$b^2 \equiv D \bmod 16$. Hence $b^2 = D + 16k$ for some k. Hence b is even say $b = 2b'$
Then we have $b'^2 = \frac{D}{4} + 4k \Rightarrow b'^2 \equiv \frac{D}{4} \bmod 4$

D fund $\Rightarrow D/4 \equiv 2, 3$ (4) but any square $0, 1 \bmod 4$.

(2) The formula also holds if $p \nmid D$ and $\left(\frac{D}{p}\right) = -1$ for some $p|n$

Since then the RHS $= \prod_{p|n}(1 + \chi_D(p)) = 0.$

and LHS $r_D(n)$ is also zero since

if $r_D(n) \neq 0$ then there are soln of

$b^2 \equiv D \mod 4n$ and this will imply a soln mod p for every $p|n$ of

$b^2 \equiv D \mod p$, ie $\left(\frac{D}{p}\right) = 1$ not -1

(3) To cover the remaining case. we note that

if $n = 2^{e_0} p_1^{e_1} \cdots p_s^{e_s}$ then

$r_D(n) = r_D(2^{e_0}) \, r_D(p_1^{e_1}) \cdots r_D(p_s^{e_s})$,

(Exercise) with $r_D(p^e) = \# \{ b \bmod p^e \mid b^2 \equiv D \ (p^e) \}$ for $p \neq 2$.

($p = 2$ case is an exercise)

Hence the proof reduces to showing that for $p|n$, p odd

$$r_D(p^e) = \begin{cases} 2 & \text{if } p \nmid D, \left(\frac{D}{p}\right) = 1 \\ 1 & \text{if } p | D, e = 1 \\ \end{cases}$$

Note $e \geq 1$ was covered in (1) since then $p^2 | n$
$p|D$

If $p|D$, and $p^2 \nmid n$, i.e. $e=1$

$\Gamma_D(p) = \#\{ b \bmod p \mid b^2 \equiv D \bmod p \}$

has only the soln $b=0$. Hence indeed

$\Gamma_D(p) = 1$. (Since $b^2 \equiv D \bmod p \Rightarrow b^2 \equiv 0 \bmod p$)
$p|D$

For the final case $p \nmid D$, $\left(\frac{D}{p}\right) = 1$

Then the set $\{ b \bmod p^e \mid b^2 \equiv D \bmod p^e \}$

has exactly 2 solns since

$b^2 \equiv D \bmod p$ has 2 solns $(b \bmod p)$

This follows from

Hensel's Lemma. Suppose p is odd prime and a is not $0 \bmod p$, $e \geq 1$ and we have a soln x_e to the eqn

$x^2 \equiv a \bmod p^e$. Then there is a unique soln x_{e+1} to the eqn

$x^2 \equiv a \bmod p^{e+1}$ which satisfies

$x_{e+1} \equiv x_e \bmod p^e$.

QED

Thm 7.8 allows us now to relate the series $\sum_{n=1}^{\infty} \frac{r_D(n)}{n^s}$ and $L(s, \chi_D)$.

We have

Prop 7.9 (1) $$\sum_{n=1}^{\infty} \frac{r_D(n)}{n^s} = \prod_p \frac{1+p^{-s}}{1-\chi_D(p)p^{-s}}$$

(2) $$\prod_p \frac{1+p^{-s}}{1-\chi_D(p)p^{-s}} = \zeta(2s)^{-1} L(s,\chi_D) . \zeta(s).$$

Proof $$r_D(n) = \begin{cases} \prod_{p|n} (1+\chi_D(p)) & \text{otherwise.} \\ 0 & \text{if } p|D,\ p^2|n \end{cases}$$

Hence $$\sum_{n=1}^{\infty} \frac{r_D(n)}{n^s} = \prod_{\substack{p \\ \left(\frac{D}{p}\right)=1 \\ (p\nmid D)}} \left(1 + \frac{2}{p^s} + \frac{2}{p^{2s}} + \cdots\right)$$

$$\prod_{\substack{p \\ \left(\frac{D}{p}\right)=-1 \\ (p\nmid D)}} (1+0\ldots) \prod_{\substack{\left(\frac{D}{p}\right)=0 \\ (p|D)}} \left(1 + \frac{1}{p^s} + \frac{0}{p^{2s}} \cdots\right)$$

Hence $\displaystyle\sum \frac{r_D(n)}{n^s} = \prod_{\substack{p \\ \left(\frac{D}{p}\right)=1}} \Big(2\underbrace{(1+p^{-s}+\cdots)}_{\frac{1}{1-p^{-s}}} - 1\Big)$

$$\prod_{-1=\left(\frac{D}{p}\right)} \left(\frac{1+p^{-s}}{1+p^{-s}}\right) \prod_{\left(\frac{D}{p}\right)=0} \left(1+p^{-s}\right)$$

$$= \prod_{\substack{p \\ \left(\frac{D}{p}\right)=1}} \left(\frac{2}{1-p^{-s}} - 1\right) \prod_{\substack{p \\ \left(\frac{D}{p}\right)=-1}} \frac{1+p^{-s}}{1+p^{-s}} \prod_{\substack{p \\ \left(\frac{D}{p}\right)=0}} 1+p^{-s}.$$

$$= \prod_{\left(\frac{D}{p}\right)=1} \frac{1+p^{-s}}{1-p^{-s}} \prod_{\left(\frac{D}{p}\right)=-1} \frac{1+p^{-s}}{1+p^{-s}} \prod_{\left(\frac{D}{p}\right)=0} \frac{1+p^{-s}}{1+0}.$$

$$= \prod_p \frac{1+p^{-s}}{1-\chi_D(p)p^{-s}}$$ Hence (1) is proved

(2) $$\prod_p \frac{1+p^{-s}}{1-\chi_D(p)p^{-s}} = \prod_p \frac{(1+p^{-s})(1-p^{-s})}{(1-\chi_D(p)p^{-s})(1-p^{-s})}$$

$$= \prod_p \frac{1-p^{-2s}}{(1-\chi_D(p)p^{-s})(1-p^{-s})} = \prod_p (1-p^{-2s}) \prod_p (1-p^{-s})^{-1} \prod_p (1-\chi_D(p)p^{-s})^{-1}$$

$$= \zeta(2s)^{-1}\zeta(s)\, L(\chi_D, s)$$

Hence $$\zeta_D(s) = \boxed{\zeta(2s) \sum \frac{r_D(n)}{n^s} = \zeta(s) L(\chi_D, s)}$$

7.43.

To prove Dirichlet's class # formula we need to understand the behaviour of $\xi_D(s) := \zeta(2s) \sum_{n=1}^{\infty} \frac{r_D(n)}{n^s}$ as $s \to 1$ which is the basis of this formula.

To this end we introduce the Epstein zeta function

<u>Defn</u> Let Q be a quadratic form in $\mathcal{Q}_D$. The <u>Epstein zeta function</u> of Q is defined as

$$Z_Q(s) := \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2}' Q(m,n)^{-s}.$$

By comparison with $\zeta(s)$ one can show that $Z_Q(s)$ converges for $\mathrm{Re}\, s > 1$.

<u>Rk</u>. Since equivalent forms take the same values if we change Q to an equivalent one in $\Gamma \backslash \mathcal{Q}_D$, we get the same sum.

Hence $Z_Q(s)$ depends only on $[Q] \in \Gamma \backslash \mathcal{Q}_D$

To see the relation of $Z_Q(s)$ to $\zeta_D(s)$

Note every $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ has a gcd that can be factored out

$$\frac{1}{2}\sum_{m,n}{}' \cdots = \frac{1}{2}\sum_{d=1}^{\infty}\sum_{(m,n)=d} Q(m,n)^{-s}$$

$$= \frac{1}{2}\sum_{d=1}^{\infty}\sum_{(m',n')=1} Q(dm',dn')^{-s}$$

$$= \frac{1}{2}\sum_{d=1}^{\infty} d^{-2s}\sum_{(m',n')=1} Q(m',n')^{-s}$$

Assume $D < -4$ for simplicity so that $w_Q = 2$

Then $$Z_Q(s) = \zeta(2s)\sum_{N=1}^{\infty}\frac{r_Q(N)}{N^s}$$

Let $Z_D(s) = \sum_{Q \in \Gamma\backslash \mathcal{Q}_D} Z_Q(s)$ Then

$$Z_D(s) = \zeta(s)\sum_{N=1}^{\infty}\left(\sum_{Q\in\Gamma\backslash\mathcal{Q}_D} r_Q(N)\right)N^{-s}$$

$$= \zeta(2s)\sum_{N=1}^{\infty}\frac{r_D(N)}{N^s}$$

Hence $\boxed{Z_D(s) = \zeta_D(s)\cdot}\ \zeta(2s)$

To study the behavour of $Z_D(s) = \zeta_D(s)$ as $s \to 1$ we can proceed in 2 dif ways:

(1) Writing $Z_Q(s)$ as Mellin transform of a theta function

(2) Recognize $Z_Q(s)$ in terms of special values of an Eisenstein series.

Both methods leads to the following thm which is the basis of class # formula

<u>Thm 7-9</u> Let $Q = [A, B, C] \in \mathcal{Q}_D$, $D < 0$

Then

$$Z_Q(s) = \frac{1}{2} \sum_{m,n \in \mathbb{Z}^2}' Q(m,n)^{-s}, \quad \operatorname{Re} s > 1$$

can be analytically continued to the whole complex s-plane with a simple pole at $s=1$ with residue $\frac{\pi}{|D|^{1/2}}$ and satisfies the

func'l eqn $\Lambda_Q(s) = \Lambda_Q(1-s)$

where

$$\Lambda_Q(s) := \left(\frac{2\pi}{|D|^{1/2}}\right)^{-s} \Gamma(s) Z_Q(s)$$

Moreover around $s=1$, $Z_Q(s)$ has the Laurent series expansion

$$Z_Q(s) = \frac{\pi}{\sqrt{|D|}} \left(\frac{1}{s-1} + C(Q) + O(s-1) \right)$$

In particular the residue $\frac{\pi}{\sqrt{|D|}}$ is indep of Q.

~~As~~ a corollary we get

Thm 7.10 (Class number formula) Let $D < -4$ be a fund. disc.

Then $$L(\chi_D, 1) = \frac{\pi h_D}{\sqrt{|D|}}.$$

Proof. $$Z_D(s) = \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} Z_Q(s) = \underbrace{\zeta(s) L(\chi_D, s)}_{= \zeta_D(s)}$$

Taking residues on both sides we have on the left $$\operatorname*{Res}_{s=1} Z_D(s) = \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \operatorname*{Res}_{s=1} Z_Q(s)$$

$$= \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \pi / \sqrt{|D|} = \frac{h_D \pi}{\sqrt{|D|}}.$$

On the right $\operatorname*{Res}_{s=1} \zeta(s) L(\chi_D, s) = L(\chi_D, 1)$

since $\zeta(s)$ has simple pole at $s=1$

Note since LHS has simple pole at $s=1$ this also ~~subject~~ shows that $L(\chi_D, s)$ does not have a pole at $s=1$.

General form of class # formula is

<u>Thm</u> $L(\chi_D, 1) = \begin{cases} \dfrac{2\pi h}{w\sqrt{|D|}} & \text{if } D<0 \\ \\ \dfrac{2h\log\varepsilon}{\sqrt{D}} & \text{if } D>0. \end{cases}$

where $\log\varepsilon = t - \sqrt{D}u/2$ coming from soln of $t^2 - Du^2 = 4$ in Thm 7.50.

We now look at

<u>Analytic properties of $Z_Q(s)$</u>:

We start with a simple observation

Given $Q = [A, B, C]$, let $z_Q = \dfrac{-B + i\sqrt{|D|}}{2A}$ $(D<0)$

Then $\|m z_Q + n\|^2 = (m z_Q + n)(m\bar{z}_Q + n)$

$$= m^2|z_Q|^2 + mn\, 2\,\mathrm{Re}\, z_Q + n^2$$

$$= m^2\left(\frac{B^2}{4A^2} + \frac{|D|}{4A^2}\right) - mn\frac{B}{A} + n^2$$

$$= m^2\left(\frac{B^2 + 4AC - B^2}{4A^2}\right) - mn\frac{B}{A} + n^2$$

$$= \frac{m^2 C}{A} - mn\frac{B}{A} + n^2 = \frac{1}{A}\left[mC^2 - mnB + An^2\right]$$

$$= \frac{1}{A} Q(n, -m) = Q(n, -m)\underbrace{\frac{\mathrm{Im}\, z_Q \cdot 2}{\sqrt{|D|}}}_{1/A}$$

Hence $\frac{\mathrm{Im}(z_Q)}{\|m z_Q + n\|^2} = \left(\frac{\sqrt{|D|}}{2}\right)\left(\frac{1}{Q(n,-m)}\right)$

Hence $Z_Q(s) = \frac{1}{2}\sum_{m,n}' Q(m,n)^{-s}$

$$= \left(\frac{|D|}{4}\right)^{-s/2} \frac{1}{2}\sum_{m,n}' \left(\frac{\mathrm{Im}\, z_Q}{\|m z_Q + n\|^{2}}\right)^{s}$$

Defn The <u>non-holomorphic Eisenstein series</u>

$$G(z,s) := \frac{1}{2}\sum_{m,n}' \frac{y^s}{|mz+n|^{2s}}$$

Then we have proved above that

<u>Lemma</u> $Z_Q(s) = \left(\frac{|D|}{4}\right)^{-s/2} G(z_Q, s)$

Hence $\boxed{\zeta_D(s) = \left(\frac{|D|}{4}\right)^{-s/2} \sum_{Q \in \Gamma\backslash \mathcal{Q}_D} G(z_Q, s)}$

Our next goal is to prove analytic properties of $G(z,s)$ from which the desired analy properties of $Z_Q(s)$ and $\zeta_D(s)$ will follow.

This can be done either by

(1) Writing the "completed zeta function

$\Lambda_Q(s) = \pi^{-s}\Gamma(s) Z_Q(s)$ as the Mellin transform

of a theta function associated to the quadratic form $Q(n,-m) = \frac{\|m z_0 + n\|^2 \sqrt{|D|}}{2\,\mathrm{Im}(z_0)}$

defined $\theta_Q(t) := \sum_{m,n}{}' e^{-\pi t Q(m,n)}$

Then Poisson summation yields

$$\boxed{\theta_Q(t) \doteq \frac{1}{t}\theta_Q(1/t)} \quad (*)$$

Then just as in the proof of A.C and F.E of $\xi(s)$ one writes

$$\pi^{-s}\Gamma(s) Z_Q(s) = \int_0^\infty \theta_Q(t) t^s \frac{dt}{t}$$

$$= \int_0^1 + \int_1^\infty$$

and using (*) and change of variables $t \to 1/t$ gives the desired A.C and F.E of $\Lambda_Q(s)$

or (2) by studying the Eisenstein series $G(z,s)$ as a modular object of wt zero and explicitly calculating its Fourier expansion

We'll follow this second path.

Thm 7.11 let $G(z,s) := \frac{1}{2} \sum' \frac{y^s}{\|mz+n\|^{2s}}$, $z = x+iy$

Then $G(z,s)$ converges for $\mathrm{Re}\, s > 1$ and for any $\gamma \in \Gamma$ we have $G(\gamma z, s) = G(z,s)$

Moreover for $\mathrm{Re}\, s > 1$ we have

$$G(z,s) = y^s \zeta(2s) + \frac{\sqrt{\pi}\, \Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta(2s-1)\, y^{1-s}$$
$$+ \frac{2\pi^s}{\Gamma(s)} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} |n|^{s-\frac{1}{2}} \sigma_{1-2s}(|n|) \sqrt{y}\, K_{s-\frac{1}{2}}(2\pi |n| y)\, e(nx)$$

Rk ① Note if write $G(z,s) = \sum_{n \in \mathbb{Z}} a_n(y,s) e(nx)$ for the F. expansion the thm says

$$a_0(y,s) = y^s \zeta(2s) + \sqrt{\pi}\, \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta(2s-1)\, y^{1-s}$$

Hence if one can show analytic properties of $G(z,s)$ without using the analy. properties of its F. coefs. This gives in fact another proof of analy. properties of Riemann zeta